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TECHNICAL REPORT No. HR-4

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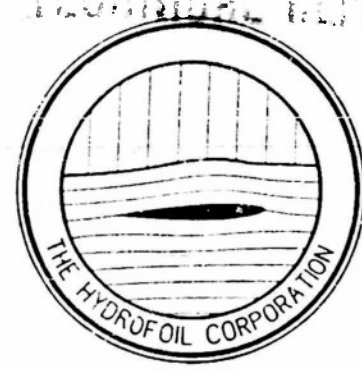
THREE-DIMENSIONAL VORTEX-LINE THEORY OF A HYDROFOIL OPERATING IN WATER OF LARGE DEPTH

Part I: The Wave Drag of Single Hydrofoil with
Prescribed Lift Distribution

Prepared for Office of Naval Research
Washington, D. C.
Contract No. 13601

February 7, 1951

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TECHNICAL REPORT HR - 4

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Operating in Water of Large Depth

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The Wave Drag of a Single Hydrofoil With Prescribed
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by

R. X. Meyer

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SUMMARY

The "bound vortex drag" of a single hydrofoil is computed for different, prescribed lift distributions. Eq. (34) gives the drag in the limiting case of a very short span; Eq. (35) in the case of a very long span.

Subsequently, the case of an arbitrary span is considered (Eq. (39)); a numerical calculation has been carried out for a Froude number $F = 10.85$ and a rectangular lift distribution (Fig. 11). Eq. (39) can be greatly simplified if the Froude number is either very small (Eq. (39^{III}) for a rectangular distribution) or very large (Eq. (39^{IV}) for a rectangular distribution).

In section 7, the problem of the optimum lift distribution of a hydrofoil is shortly discussed in a qualitative way.

THE HYDROFOIL CORPORATION

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References:

- (a) M. H. Vavra: "Wave Drag of Submerged Foils in Shallow Water", Technical Memorandum HM - 3, The Hydrofoil Corporation, November 19, 1950.
- (b) G. P. Weinblum: "Analysis of Wave Resistance", Taylor Model Basin Report 710, 1950.
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- (d) Lamb: "Hydrodynamics", 5th. Ed., Cambridge, 1930.
- (e) Prandtl-Tjetjens: "Applied Hydro- and Aerodynamics", McGraw-Hill, 1st. Ed., 1934.
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- (h) Watson: "Bessel Functions", 2nd. Ed., Cambridge, 1945.
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1. Introduction

1. A hydrofoil of semi-span b is considered, moving at a constant speed of advance V and constant submergence through water of infinite depth (Fig. 1). The water is assumed to be undisturbed except by the action of the hydrofoil itself.

In the present approach, for the purpose of a simplified analysis, the actual foil is replaced by a single, straight "bound - vortex-line" in conformity with Prandtl's airplane wing theory. The bound - vortex-line is taken parallel to the (undisturbed) interface, at a distance f from it, and located approximately at the centers of lift of the wing sections of the hydrofoil.

An arbitrary spanwise lift-distribution is prescribed. The problem is similar, therefore, to the so-called "first problem of airfoil theory", complicated here by the presence of an interface.

Viscous forces are neglected. Irrotational fluid motion is assumed everywhere outside the bound vortex and the trailing vortex sheet.

The coordinate system $x-y-z$ in Fig. 1 is considered as being fixed with the foil. The $x-y$ plane coincides with the undisturbed position of the interface. Viewed in this frame, the fluid-motion is steady. The velocity at infinity, V , shall have the direction of the positive x axis.

2. The velocity at any given point of the fluid is designated by the vector \vec{c} in the $x-y-z$ system. It will be assumed that, for points on the free surface,

$$\frac{|\vec{c} - \vec{V}|}{V} \ll 1$$

(1)

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such that terms of the order of magnitude of $\left(\frac{\tilde{c} - \tilde{v}}{v}\right)^2$ may be neglected. The above assumption is the better fulfilled, in the case of similar lift-distributions, the smaller

$$\frac{\Gamma_{max}}{V f}$$

where Γ_{max} is the maximum value of the spanwise circulation-distribution. Numerical examples in Refs. (a) and (c) indicate that Eq. (1) should be well fulfilled in most actual applications of hydrofoils.

It follows from the above postulate that the slope of the surface waves and their height are assumed to be small. The latter statement follows from an application of Bernoulli's equation to the free surface. In fact, it is found that the wave height is, by virtue of Eq. (1), assumed to be small compared with the deep-water wavelength $2\pi V^2/g$ (e.g. Ref. (d) § 228).

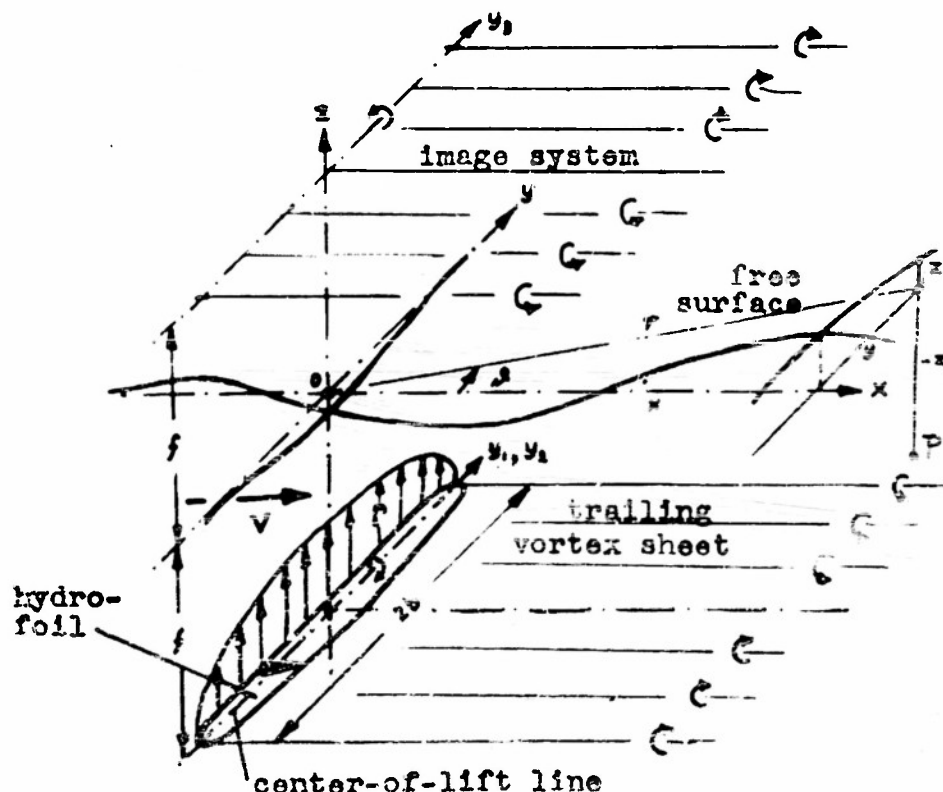


Fig.1

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Also, it will be assumed that the wave height is small compared with the submergence of the foil.

Due to these simplifying assumptions the boundary condition at the free surface is linear (section 3.3).

3. Strictly speaking, the trailing vortices coincide with streamlines. In the case of a wing in an infinite medium, however, it is known that a good approximation to the induced drag is obtained, if the trailing vortex lines are assumed to be straight and in the direction of the free stream velocity. At least this is the case if the induced drag over lift ratio is small, which in turn requires a reasonably high aspect ratio.

In the case of a hydrofoil, the presence of the interface causes an additional distortion of the trailing vortices. The vortex sheet follows now to a certain extent the wave motion of the surface. Due to the damping effect exerted by the depth, the amplitude of this motion is diminished as compared with the one of the surface. It follows from the conclusions drawn in section 1.2, that this distortion of the trailing vortices is small. One neglects only small terms of higher order if one assumes, for the purpose of computing the wavedrag, that the trailing vortex lines are straight. With this assumption, the boundary condition for the velocity at the vortex sheet is linear.

Concluding, one may note that the (in-viscous) drag is small compared with the lift in nearly all cases of technical interest. This means that the downwash is small compared with the speed of advance, which indicates that the linearization of the boundary condition at the surface and at the vortex sheet is permissible in these cases.

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2. Nomenclature

a, Aconstants (Eqs. (26"), (17:))
 bsemi-span
 C_Llift coefficient of foil
 D_Bbound vortex drag
 D_Ttrailing vortex drag
 fsubmergence
 FFroude number (Eq. (33))
 gacceleration due to gravity
 Llift
 L_ilift per unit span
 nconstant (section 7.2)
 r, Rradii (Figs. 1, 6)
 Swing area
 ttime
 Vspeed of advance
 wdownwash in y - z plane, induced by image of wing-element
 $x, y, z, y_1, y_2, y_3, \dots$coordinates (Fig. 1)
 z_selevation of free surface
 βspan-submergence ratio (Eq. (38))
 γangle (Fig. 9)
 ΓCirculation
 ΓGamma function
 \mathcal{J}angular coordinate (Fig. 1)
 θ, θ'angular coordinates (Figs. 6, 9)
 μconstant (Eq. (35))
 ρDensity

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ϕ perturbation potential pertaining to wing element

$\bar{\phi}$ perturbation potential of rotationally symmetric problem (section 4.1)

$\Psi_0, \Psi_1, \Psi_2, \Psi_3$ dimensionless functions (Eqs. (34), (35), (36), (39))

$-$ superscript indicating spanwise average.

3. Basic Equations

1. It is convenient to consider the velocity with respect to the x-y-z coordinate system, at any point below the interface, as the resultant of the following velocities:

- (1) The free stream velocity \vec{V} in the positive x direction.
- (2) The velocity induced by
 - (a) the bound vortex,
 - (b) the trailing vortexes.
- (3) The velocity induced by
 - (a) an image of the bound vortex, i.e. the vortex obtained by reflecting the bound vortex at the x-y plane and changing its sense of rotation (Fig. 1),
 - (b) the corresponding images of the trailing vortexes.
- (4) The velocity derived from a perturbation potential $\phi(x, y, z)$.

This merely amounts to a definition of ϕ , the determination of which will be the first problem considered.

At all points below the interface, items (2), (3), and (4) are of the nature of a small perturbation added to \vec{V} . The only exception to this arises in the immediate neighbourhood of the foil, where item (2a) assumes large values.

The velocity defined in (4) above is irrotational everywhere, including points situated on the bound - and trailing vortex lines. Hence, the

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resultant velocity obtained by addition of (1), (2), (3), and (4), is also irrotational, as is required, except at points located on the bound and trailing vortexes of the foil, where, due to item (2), the correct circulation is obtained.

Item (3) above is introduced merely for mathematical convenience (section 3.3). Omitting or adding it amounts only to different definitions of ϕ .

2. We consider first the case where the span $2b$ is very small compared with the other characteristic dimensions of the problem *). The general case of a finite span is then obtained by superposition of the results derived for the infinitesimal span.

3. There is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (2)$$

the solution being subject to the boundary conditions derived in this section.

Neglecting terms of higher order, one may replace the boundary conditions at the interface by conditions imposed at $z = 0$. We examine first the velocity induced at any point of the x - y plane, by items (2) and (3) of section (3.1).

The bound vortex (2a) and its image (3a) induce together a velocity, the y and z components of which, vanish **) by reason of symmetry. The x component becomes, for $2b \ll f$

*) The quantity V^2/g , which enters the Froude number, has the dimension of a length and, in general, must also be considered as a characteristic dimension of the problem. The same applies to Γ/V , a quantity which in the case where the foil is replaced by a vortex line, appears in lieu of the chord. Actually, however, it is shown in a subsequent part of the analysis (section 7) that the above conditions may be replaced by a less stringent one: It is sufficient to require $2b \ll f$ in order that the equations of the present section be valid.

**) The fact that the z component vanishes, is the main advantage gained by the introduction of the image.

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$$\frac{b\bar{\Gamma}f}{\pi} (r^2 + f^2)^{-3/2}$$

from the analogon to the "Law of Biot and Savart" in electrodynamics.

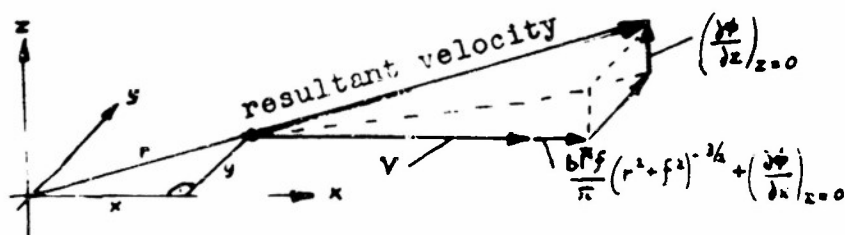
There is $r^2 = x^2 + y^2$ and $\bar{\Gamma}$ denotes the average circulation, that is

$$\bar{\Gamma} = \frac{1}{2b} \int_{-b}^{+b} \Gamma(y) dy,$$

The trailing vortexes (2b) and their images (3b) induce together a velocity with vanishing x and z components. It will appear that the y components of velocities do not enter explicitly the boundary conditions for the free surface; there is no need therefore for an expression for the velocity induced by the trailing vortexes together with their images.

Fig. 2 depicts the resultant velocity and its components at a point located on the x-y plane.

Fig. 2



The pressure is constant for points on the free surface and may be put equal to zero without loss of generality. Hence, from Bernoulli's equation, neglecting terms of higher order in the perturbation velocity,

$$\frac{\rho}{2} \left\{ V^2 + 2V \left[\frac{b\bar{\Gamma}f}{\pi} (r^2 + f^2)^{-3/2} + \left(\frac{\partial\phi}{\partial x} \right)_{z=0} \right] \right\} + \rho g z_0 = \frac{\rho}{2} V^2$$

since $z_0 = 0$ for $x = -\infty$. $z_0 = z_0(x, y)$ is the local elevation of the free surface from its undisturbed position.

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Simplifying the preceding equation, one obtains

$$z_0 = -\frac{V}{g} \left[\frac{b\bar{r}f}{\pi} (r^2 + f^2)^{-3/2} + \left(\frac{\partial \phi}{\partial x} \right)_{z=0} \right]; \quad 2b \ll f \quad (3)$$

The constant-pressure surface $z_0(x,y)$ as defined by Eq. (3), must be tangent everywhere to the velocity, since it is at the same time a fluid boundary. Neglecting again terms of higher order, this condition becomes,

$$\frac{\partial z_0}{\partial x} = \frac{1}{V} \left(\frac{\partial \phi}{\partial x} \right)_{z=0} \quad (4)$$

Eliminating z_0 by differentiation with respect to x of Eq. (3) and inserting into Eq. (4), one obtains the following boundary condition for ϕ ,

$$\left(\frac{\partial^2 \phi}{\partial x^2} \right)_{z=0} + \frac{g}{V^2} \left(\frac{\partial \phi}{\partial z} \right)_{z=0} - \frac{3b\bar{r}f}{\pi} (r^2 + f^2)^{-5/2} x = 0 \quad (5)$$

Assuming infinite depth of the water, one has the additional boundary conditions,

$$\left(\frac{\partial \phi}{\partial x} \right)_{z=-\infty} = \left(\frac{\partial \phi}{\partial y} \right)_{z=-\infty} = \left(\frac{\partial \phi}{\partial z} \right)_{z=-\infty} = 0 \quad (6)$$

since the disturbance produced by the foil must vanish at large depth.

Finally, since we assume that the water is undisturbed except by the action of the foil itself,

$$\left(\frac{\partial \phi}{\partial x} \right)_{x=-\infty} = \left(\frac{\partial \phi}{\partial y} \right)_{x=-\infty} = \left(\frac{\partial \phi}{\partial z} \right)_{x=-\infty} = 0 \quad (7)$$

Eq. (2) together with the boundary conditions (5), (6), and (7) determine the potential ϕ .

4. Before proceeding with the equations so far derived, it is interesting to discuss a problem which is closely related to the present one.

The term $\frac{b\bar{r}f}{\pi} (r^2 + f^2)^{-3/2}$ in the equation preceding Eq. (3) did arise from a velocity component (Fig. 2), but it could also have

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arisen from a pressure distribution imposed on the free surface *).

This suggests to consider the surface wave pattern produced by an air-plane flying at an altitude f' over calm water, at a horizontal speed V' (sea plane at take-off). If we again first assume that the span $2b'$ is small compared with f' , one can show (e.g. Ref. (e)) that a pressure increment

$$p' = \rho' V' \frac{b' \bar{\Gamma} f'}{\pi} (r'^2 + f'^2)^{-3/2} \quad (8)$$

results at the surface. ρ' here is the density of the air. Eq. (8) is usually derived for a plane, rigid boundary. Taking into account the large difference in density of air and water, it is not difficult to show that one neglects only terms of higher order if Eq. (8) is applied instead to a free water surface. Indeed, this is also obvious intuitively.

The velocity of the water, relative to a coordinate system x', y', z' , fixed with the airplane, can be taken as the resultant of the following velocities:

- (1) The free stream velocity \vec{V}' in the positive x' direction.
- (2) The velocity derived from a perturbation potential, $\phi'(x', y', z')$.

There is,

$$\frac{\partial \phi'}{\partial x'^2} + \frac{\partial \phi'}{\partial y'^2} + \frac{\partial \phi'}{\partial z'^2} = 0 \quad (9)$$

Bernoulli's equation, applied to the water-side of the interface becomes

$$\rho' V' \frac{b' \bar{\Gamma} f'}{\pi} (r'^2 + f'^2)^{-3/2} + \frac{\rho'}{2} \left\{ V'^2 + 2V' \left(\frac{\partial \phi'}{\partial x'} \right)_{z'=0} \right\} + \rho' g z' = \frac{\rho'}{2} V'^2$$

or

$$z' = - \frac{V'}{g} \left[\frac{\rho'}{\rho} \frac{b' \bar{\Gamma} f'}{\pi} (r'^2 + f'^2)^{-3/2} + \left(\frac{\partial \phi'}{\partial x'} \right)_{z'=0} \right]$$

With

$$\frac{\partial z'_0}{\partial x'} = \frac{1}{V'} \left(\frac{\partial \phi'}{\partial x'} \right)_{z'=0}$$

*) Compare also Ref. (f)

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this becomes,

$$\left(\frac{\partial \phi'}{\partial x'}\right)_{x'=0} + \frac{g}{V'^2} \left(\frac{\partial \phi'}{\partial z'}\right)_{x'=0} - \frac{\rho'}{\rho} \frac{3b'\bar{\Gamma}'f'}{\pi} (r'^2 + f'^2)^{-5/2} x' = 0 \quad (10)$$

Also

$$\left(\frac{\partial \phi'}{\partial x'}\right)_{x'=-\infty} = \left(\frac{\partial \phi'}{\partial y'}\right)_{x'=-\infty} = \left(\frac{\partial \phi'}{\partial z'}\right)_{x'=-\infty} = 0 \quad (11)$$

and

$$\left(\frac{\partial \phi'}{\partial x'}\right)_{x'=-\infty} = \left(\frac{\partial \phi'}{\partial y'}\right)_{x'=-\infty} = \left(\frac{\partial \phi'}{\partial z'}\right)_{x'=-\infty} = 0 \quad (12)$$

Comparison of Eqs. (2), (5), (6), and (7) for ϕ with Eqs. (9), (10), (11), and (12) for ϕ' shows that

$$\phi(x, y, z) = \frac{L}{L'} \left(\frac{V'}{V}\right)^3 \phi'(x', y', z') \quad (13)$$

provided

$$\frac{V^3}{fg} = \frac{V'^3}{f'g'} \quad (14)$$

and

$$\frac{x}{f} = \frac{x'}{f'} ; \quad \frac{y}{f} = \frac{y'}{f'} ; \quad \frac{z}{f} = \frac{z'}{f'} \quad (15)$$

where $L = 2b\rho V\bar{\Gamma}$ and $L' = 2b'\rho'V'\bar{\Gamma}'$ are the lift of the hydrofoil and the airplane wing respectively.

From Eq. (3) then follows

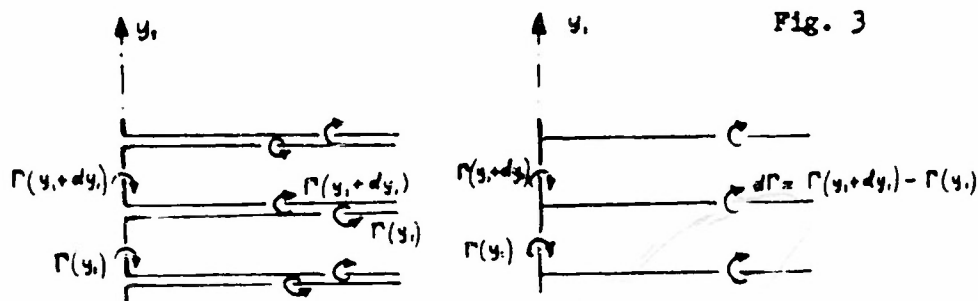
$$z_0(x, y) = \frac{L}{L'} \left(\frac{f'}{f}\right)^2 z_0'(x', y') = \frac{L}{L'} \left(\frac{V'}{V}\right)^4 z_0'(x', y') \quad (16)$$

Since the equations determining the potential are linear, the case of a finite span can be obtained by superposition of the effects of small wing elements. Wing elements of rectangular lift-distribution can be chosen and the limiting process employed, which is schematically indicated in Fig. 3.

Since superposition is permissible, the results so far derived must be valid also for the finite span and may be formulated as follows:

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Provided that a hydrofoil and an aircraft wing are geometrically similar and have the same Froude number (Eq. 14), the perturbation potential ϕ and the surface disturbance z .



are similar in both cases, in the sense that Eqs. (13) and (16) hold. The factor of proportionality in Eq. (16) being a positive quantity, depressions of the water surface in one case are also depressions in the other, etc.

For instance, with $L' = 13000$ lb, $2b' = 70$ ft., $V' = 120$ m.p.h. (Grumman G-73 Flying-boat) and $f' = 20$ ft., the dimensions of the corresponding hydrofoil craft would become, with $V = 40$ knots: $2b = 10.4$ ft., $f = 3$ ft. Assuming $L = 1000$ lb, z_0' is 29% of z_0 .

4. Solution For The Potential ϕ

1. The fact that the inhomogeneous part

$$= \frac{V}{g} \cdot \frac{b \Gamma f}{\pi} (r^2 + f^2)^{-3/2}$$

of Eq. (3) is rotationally symmetric with respect to the z -axis, suggests to deal first with a modified problem *) which is entirely symmetric with respect to the z -axis: Assume calm water and a coordinate system at rest relative to

*) The approach here is the same as used by Lamb (Ref. (d)) and in various papers by Havelock.

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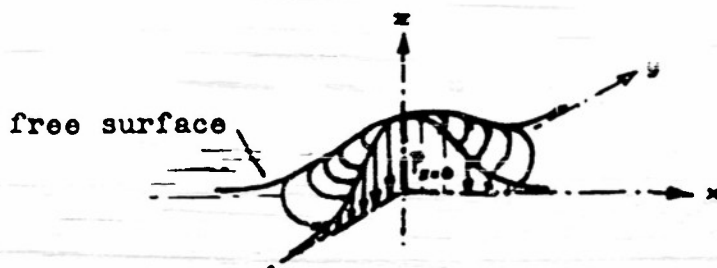
the water (Fig. 4). An external pressure $P_{z..}$ distributed over the free surface according to

$$P_{z..} = \varphi A (r^2 + f^2)^{-\frac{1}{2}} \quad (17)$$

shall be suddenly applied and maintained during a short interval δt , after which it is removed ("impulsive pressure"). A and f above are constants. In the process of building up the solution for the submerged foil from the solution of the modified problem presently considered, the constant f above will be taken equal to the submergence of the foil, whereas A will be taken

$$A = \frac{v b \bar{r} f}{\pi} \quad (17')$$

(section 3.4).



The effect of the externally applied pressure is "felt" instantaneously at any point within the (incompressible) fluid. At any such point, defined by coordinates r, z , a pressure $P(r, z)$ results, in addition to the hydrostatic pressure due to gravity. $P(r, z)$ is constant during the time interval δt and fulfills the equation

$$\nabla^2 P = 0$$

together with the boundary condition Eq. (17) at $z = 0$ (e.g. Ref. (g), § 73).

At the end of the interval, the fluid has acquired a velocity which can be derived from a potential, which in turn is equal to $-\frac{\delta t}{\rho} P$ (Ref. (g)).

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The motion which takes place after the time $t = 0$, that is after the external pressure has been removed at the end of the interval δt , can be described by a potential $\Phi(r, z, t)$, which fulfills the equation

$$\nabla^2 \Phi = 0 \quad (18)$$

Since, $\Phi_{t=0} = -\frac{\delta t}{\rho} P$, one obtains from Eq. (17) the condition

$$\Phi_{z=0} = -A \delta t (r^2 + f^2)^{-1/2} \quad (18')$$

For the elevation $Z_0(r, t)$ of the free surface one has the initial condition

$$(Z_0)_{t=0} = 0$$

since at the end of the time interval δt , Z_0 can be shown to be of the order of $(\delta t)^2$ and consequently is neglected. Capital letters are used in order to distinguish the various functions pertaining to the present problem from those pertaining to the case of the submerged foil.

Applying Bernoulli's equation to the free surface, at a time $t > 0$ and neglecting the square of the velocity as of higher order,

$$g Z_0 + \left(\frac{\partial \Phi}{\partial t} \right)_{z=0} = 0$$

(The arbitrary time function which is sometimes retained in the formulation of Bernoulli's equation for instationary flow can be merged into the potential).

Furthermore one has to satisfy at the surface the kinematical condition,

$$\frac{\partial Z_0}{\partial t} = \left(\frac{\partial \Phi}{\partial z} \right)_{z=0}$$

if terms of higher order are again neglected. Eliminating Z_0

from the last two equations, one obtains as boundary conditions for Φ ,

$$\left(\frac{\partial^2 \Phi}{\partial t^2} \right)_{z=0} + g \left(\frac{\partial \Phi}{\partial z} \right)_{z=0} = 0 \quad (18'')$$

Also,

$$\left(\frac{\partial \Phi}{\partial r} \right)_{r=\infty} = \left(\frac{\partial \Phi}{\partial z} \right)_{z=\infty} = 0 \quad (18''')$$

and

$$\left(\frac{\partial \Phi}{\partial r} \right)_{r=0} = \left(\frac{\partial \Phi}{\partial z} \right)_{z=0} \quad (18''')$$

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2. A fundamental solution of Eq. (18) is

$$\bar{F} = e^{kz} J_0(kr) ; \quad k \geq 0$$

where J_0 is the zero order Bessel function and k an arbitrary constant. From the principle of superposition of particular solutions of linear equations,

$$\bar{F}(r, z, t) = \int_0^\infty \alpha(k) \cos(\sqrt{gk} t) e^{kz} J_0(kr) dk \quad (19)$$

must also be a solution. $\alpha(k)$ is a function of k to be determined later.

Assuming that the integral in Eq. (19) is uniformly convergent and differentiating under the integral sign, one shows easily that Eq. (18) is fulfilled by the expression (19). Similarly, Eqs. (18)^{II} and (18)^{IV} can be shown to be satisfied. For instance, there is

$$\left(\frac{\partial \bar{F}}{\partial r} \right)_{r=\infty} = - \left(\int_0^\infty k \alpha(k) \cos(\sqrt{gk} t) e^{kz} J_1(kr) dk \right)_{r=\infty} = 0$$

as required by the first one of Eqs. (18)^{IV}.

Finally, substituting expression (19) into the remaining boundary condition Eq. (18)^I,

$$\int_0^\infty \alpha(k) J_0(kr) dk = -A \delta t (r^2 + f^2)^{-3/2}$$

In order to determine $\alpha(k)$ from this equation, we make use of the integral theorem for Bessel functions,

$$F(r) = \int_0^\infty k J_0(kr) dk \int_0^\infty l \bar{F}(l) J_0(kl) dl$$

Taking here for the function $F(r)$

$$F(r) = A \delta t (r^2 + f^2)^{-3/2}$$

one has

$$\int_0^\infty \alpha(k) J_0(kr) dk = -A \delta t \int_0^\infty k J_0(kr) dk \int_0^\infty l (l^2 + f^2)^{-3/2} J_0(kl) dl \quad (20)$$

The last integral occurring here is brought with the substitution of the

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new variable $m = \frac{l}{f}$ for l on a standard form,

$$\int_0^\infty l(l^2 + f^2)^{-1/2} J_0(kl) dl = \frac{1}{f} \int_0^\infty m(m^2 + 1)^{-1/2} J_0(cm) dm$$

where $c = fk \geq 0$. The integral on the right is simply

$$\int_0^\infty m(m^2 + 1)^{-1/2} J_0(cm) dm = e^{-c}$$

which is a special case of one of Sonine's formulas (Ref. (h), § 13.6).

Hence, from Eq. (20),

$$\int_0^\infty \alpha(k) J_0(kr) dk = - \frac{A \delta t}{f} \int_0^\infty k e^{-fk} J_0(kr) dk$$

which is satisfied by

$$\alpha(k) = - \frac{A \delta t}{f} k e^{-fk} \quad (21)$$

Consequently, one has solution of Eqs. (18) to (18^{IV}), from Eq. (19),

$$\Phi(r, z, t) = - \frac{A \delta t}{f} \int_0^\infty \cos(\sqrt{r^2 + z^2} k) k e^{k(z-f)} J_0(kr) dk \quad (22)$$

3. The case of a hydrofoil of small span $2b \ll f$ is now obtained from the results of the previous section by a process of superposition *).

Assume, as before, that the pressure forces Eq. (17) act on the free surface during a time interval δt . The waves produced by this process are described by Eq. (22). At the end of the interval δt , suppose that the process be repeated, but now with the center of the pressure distribution at a distance $V \cdot \delta t$ from the original center, and so forth. In the limit, for vanishing δt , one obtains the case of a pressure distribution moving continuously at a speed V . The potential function pertaining to this case is identical to the one pertaining to the hydrofoil (section 3.4).

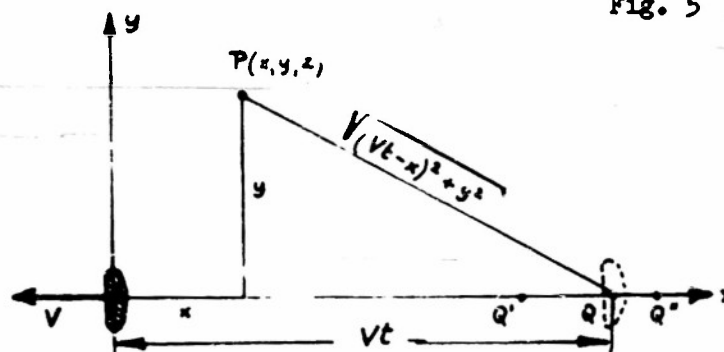
Consider a point P with coordinates x, y, z in a coordinate system fixed with the foil (Fig. 5). Also consider a point Q through which the foil has passed t units of time earlier; Q is then at a horizontal distance Vt behind the foil. At the present moment, P receives waves

*) Compare footnote p. 12

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which originated at Q, t units of time earlier. Similarly it receives waves from points Q', Q'', etc., though the time elapsed is different. Replacing therefore in Eq. (22) r by $\sqrt{(Vt-x)^2 + y^2}$ and integrating over t from 0 to ∞ , one obtains for the potential ϕ pertaining

Fig. 5



to the hydrofoil of small span,

$$\phi(x, y, z) = - \frac{Vb\Gamma}{\pi} \int_{t=0}^{\infty} \int_{k=0}^{\infty} \cos(\sqrt{gk}t) k e^{k(z-f)} J_0(k\sqrt{(Vt-x)^2 + y^2}) dt dk \quad (23)$$

$2b \ll f$

The above expression for ϕ satisfies all the conditions (Eqs. (2), (5), (6), and (7)) imposed on ϕ . In order to check this in the case of Eq. (5), for instance, we have to find first $\frac{\partial \phi}{\partial x}$. For convenience, we put

$$k\sqrt{(Vt-x)^2 + y^2} = r$$

from which

$$\frac{\partial r}{\partial x} = -V \frac{\partial r}{\partial t}$$

Differentiating (23) under the integral sign and altering the sequence of integration,

$$\frac{\partial \phi}{\partial x} = + \frac{Vb\Gamma}{\pi} \int_{k=0}^{\infty} k e^{k(z-f)} dk \int_{t=0}^{\infty} \cos(\sqrt{gk}t) J_1(r) \frac{\partial r}{\partial x} dt$$

The last integral, if integrated by parts, is

$$- \frac{1}{V} J_0(kr) + \frac{\sqrt{gk}}{V} \int_{t=0}^{\infty} \sin(\sqrt{gk}t) J_0(r) dt$$

Making use of the relationship

$$\int_{k=0}^{\infty} k e^{k(z-f)} J_0(kr) dk = (f-z) \sqrt{r^2 + (f-z)^2}^{-\frac{3}{2}}$$

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which occurs frequently in applications of Bessel functions*) one obtains

$$\frac{\partial \phi}{\partial x} = - \frac{b\bar{\Gamma}(f-z)}{\pi} (r^2 + (f-z)^2)^{-3/2} + \frac{b\bar{\Gamma}g}{\pi} \int_{t=0}^{\infty} \int_{k=0}^{\infty} \sin(\sqrt{gk}t) k^{3/2} e^{k(z-f)} J_0(k\sqrt{(vt-x)^2 + y^2}) dt dk \quad (23^I)$$

Differentiating and integrating by parts again, leads to

$$\frac{\partial^2 \phi}{\partial x^2} = + \frac{3b\bar{\Gamma}(f-z)}{\pi} (r^2 + (f-z)^2)^{-5/2} + \frac{b\bar{\Gamma}g}{\pi V} \int_{t=0}^{\infty} \int_{k=0}^{\infty} \cos(\sqrt{gk}t) k^2 e^{k(z-f)} J_0(k\sqrt{(vt-x)^2 + y^2}) dt dk \quad (23^{II})$$

Upon substitution into the left of Eq. (5), together with

$$\frac{\partial \phi}{\partial z} = - \frac{Vb\bar{\Gamma}}{\pi} \int_{t=0}^{\infty} \int_{k=0}^{\infty} \cos(\sqrt{gk}t) k^2 e^{k(z-f)} J_0(k\sqrt{(vt-x)^2 + y^2}) dt dk \quad (23^{III})$$

one recognizes that Eq. (5) is indeed satisfied.

The surface elevation is now obtained from Eqs. (3) and (23^I),

$$z_s(x,y) = - \frac{Vb\bar{\Gamma}}{\pi Vg} \int_{t=0}^{\infty} \int_{k=0}^{\infty} \sin(\sqrt{gk}t) k^{3/2} e^{-fk} J_0(k\sqrt{(vt-x)^2 + y^2}) dt dk \quad 2b \ll f \quad (24)$$

5. The Wave Drag of a Hydrofoil of Small Span

1. The following considerations are valid for the general case, where the span may be arbitrarily large; later we shall restrict our attention to the case $2b \ll f$.

Neglecting viscosity, the drag of a foil can be obtained either by computing the "downwash" at the foil, or by considering the energy gained by the fluid in the rear of the foil. We shall adopt here the method mentioned first.

At any point on the center of lift line of the foil, the downwash is composed of the contributions made by items (2b), (3b), and (4) of section 3.1. The downwash produced by (2b), and (3b) can be obtained from results derived in the wing theory of airplanes. The downwash resulting from (4) will be obtained by a superposition of the result obtained in Eq. (23^{III}) valid in the case $2b \ll f$.

To each component of the downwash corresponds a component of the drag. The total drag may then be separated into the four (additive) components listed in table 1.

*) The same relationship was used on pages 15/16.

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Table I

	Corresponding Downwash	Remarks
(1) Trailing Vortex Drag *) D_T	Induced by trailing vortexes, (2b) of section II.3.1.	Identical with the induced drag of a mono-plane. Independent of submergence.
(2) Bound Vortex Drag *) D_B	(a) Induced by trailing vortexes of image, (3b) of section II.3.1.	Identical with the "ground effect" of an airplane. This component of the drag is negative.
	(b) Velocity derived from perturbation potential, (4) of section II.3.1.	
(3) Profile Drag		
(4) In cases to which the present theory is not fully applicable, additional drag may be experienced. To those cases belongs the formation of shock fronts, which cannot be rendered by a theory utilising linearised boundary conditions.		

*) These expressions were first proposed in Ref. (a)

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Since the downwash pertaining to the bound vortex drag can also be regarded as being "induced", it seems preferable to use the expression "trailing vortex drag" rather than "induced drag" for the first item in table 1 *). In the case of an infinite span and constant circulation only a bound vortex is present and only item (2) (besides (3) and possibly (4)) gives a contribution to the drag. Hence, it seems justified to apply the expression "bound vortex drag" to item (2); and this also in the case of a finite span.

Since the trailing vortex drag is obtained from the well-known expressions developed for the induced drag of an airplane wing, we can now restrict our attention to the bound vortex drag D_b .

2. Consider first the downwash w at any point P in the y - z plane, produced by the trailing vortexes image of a wing of small span (Fig. 6).

The trailing vorticity shed between y_1 and $y_1 + dy_1$ induces at P a velocity dw^* in the y - z plane. For the integrated effect of this trailing vortex, extending from $x = 0$ to $x = \infty$, one finds (e.g. Ref. (a)),

$$dw^* = - \frac{d\Gamma}{4\pi R'}$$

Neglecting terms of higher order,

$$R' = R - y_1 \sin \theta$$

from Fig. 6. Hence,

$$dw^* = - \frac{d\Gamma}{4\pi R} \left(1 + \frac{y_1}{R} \sin \theta \right)$$

*) Ref. (a).

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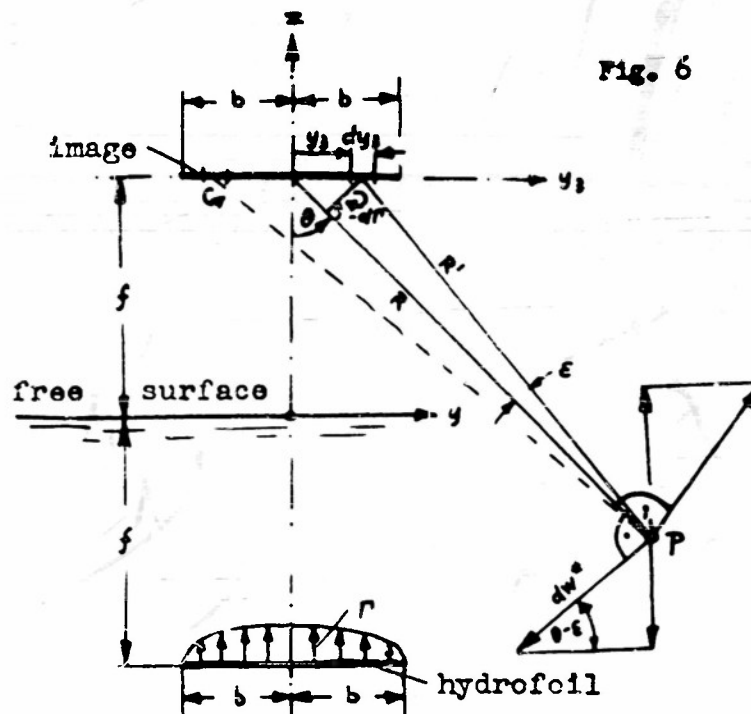


Fig. 6

if higher order terms are again neglected. With

$$\epsilon = \frac{y_3 \cos \theta}{R}$$

$$\sin(\theta - \epsilon) = \sin \theta - \epsilon \cos \theta = \sin \theta \left(1 - \frac{y_3}{R} \frac{\cos^2 \theta}{\sin \theta} \right)$$

the vertical component of the induced velocity is

$$\frac{\sin \theta}{4\pi R} \frac{d\Gamma}{dy_3} \left[1 + \frac{y_3}{R} \left(\sin \theta - \frac{\cos^2 \theta}{\sin \theta} \right) \right] = \frac{\sin \theta}{4\pi R} \frac{d\Gamma}{dy_3} \left[1 - \frac{y_3}{R} \frac{\cos(2\theta)}{\sin \theta} \right]$$

Similarly, the vertical component of the velocity induced by the

trailing vortex shed at $-y_3$ is

$$= \frac{\sin \theta}{4\pi R} \frac{d\Gamma}{dy_3} \left[1 + \frac{y_3}{R} \frac{\cos(2\theta)}{\sin \theta} \right]$$

Adding the last two equations and integrating from 0 to b ,

one has for the downwash w at P , produced by the trailing vortices image,

$$w = \frac{\cos(2\theta)}{2\pi R^2} \int_0^b y_3 \frac{d\Gamma}{dy_3} dy_3$$

if the downwash is considered as positive if directed downwards.

Through integration by parts, one has finally,

$$w = - \frac{b\Gamma \cos(2\theta)}{2\pi R^2} \quad ; \quad \left. \begin{array}{l} 2b \ll f \\ x = 0 \end{array} \right\} \quad (25)$$

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3. Next we shall find a closed expression for $\frac{\partial \phi}{\partial z}$ in terms of tabulated functions, for points in the lower half of the y-z plane.

From Eq. (23^{III}) with $X = 0$ and the substitution $\tau = Vt$,

$$\frac{\partial \phi}{\partial z} = - \frac{b\Gamma}{\pi} \int_{k=0}^{\infty} k^2 e^{k(z-f)} dk \int_{\tau=0}^{\infty} \cos\left(\sqrt{\frac{gk}{V^2}} \tau\right) J_0(k\sqrt{\tau^2+y^2}) d\tau$$

The second integral here is a Sonine discontinuous integral and has the value (Ref. (h), §13.47)

$$\int_{\tau=0}^{\infty} \cos\left(\sqrt{\frac{gk}{V^2}} \tau\right) J_0(k\sqrt{\tau^2+y^2}) d\tau = \begin{cases} 0 & \text{for } k < \sqrt{\frac{gk}{V^2}} \text{ i.e. } k < \frac{g}{V^2} \\ \frac{\cos\left(y\sqrt{k^2 - \frac{gk}{V^2}}\right)}{\sqrt{k^2 - \frac{gk}{V^2}}} & \text{for } k > \frac{g}{V^2} \end{cases}$$

It is therefore sufficient to begin the integration over k at a lower limit $k = g/V^2$ rather than at $k = 0$. Hence,

$$\frac{\partial \phi}{\partial z} = - \frac{b\Gamma}{\pi} \int_{k=g/V^2}^{\infty} \frac{k^2 e^{k(z-f)} \cos\left(y\sqrt{k^2 - \frac{gk}{V^2}}\right)}{\sqrt{k^2 - \frac{gk}{V^2}}} dk$$

The presence of the square root suggests to substitute a new variable of integration κ by

$$k = \frac{g}{V^2} \cosh^2\left(\frac{\kappa}{2}\right) = \frac{g}{2V^2} (\cosh \kappa + 1)$$

from which

$$\sqrt{k^2 - \frac{gk}{V^2}} = \frac{g}{V} \cosh\left(\frac{\kappa}{2}\right) \sinh\left(\frac{\kappa}{2}\right) = \frac{g}{2V^2} \sinh \kappa$$

It is convenient to introduce again the polar coordinates R, θ (Fig. 6)

where

$$z - f = -R \cos \theta$$

$$y = R \sin \theta$$

Consequently,

$$\frac{\partial \phi}{\partial z} = - \frac{g^2 b \Gamma}{4\pi V^4} e^{-\frac{gR}{2V^2} \cos \theta} \int_{\kappa=0}^{\infty} (\cosh \kappa + 1)^2 e^{-\frac{gR}{2V^2} \cos \theta \cosh \kappa} \cos\left(\frac{gR}{2V^2} \sin \theta \sinh \kappa\right) d\kappa$$

With

$$\cos\left(\frac{gR}{2V^2} \sin \theta \sinh \kappa\right) = \operatorname{Re} e^{-i\left(\frac{gR}{2V^2} \sin \theta \sinh \kappa\right)}$$

(Re = real part) and simplifying,

$$\frac{\partial \phi}{\partial z} = - \frac{g^2 b \Gamma}{4\pi V^4} e^{-\frac{gR}{2V^2} \cos \theta} \operatorname{Re} \int_{\kappa=0}^{\infty} (\cosh \kappa + 1)^2 e^{-\frac{gR}{2V^2} \cos(\theta - i\kappa)} d\kappa$$

or

$$\frac{\partial \phi}{\partial z} = - \frac{g^2 b \Gamma}{4\pi V^4} e^{-\alpha \cos \theta} \left[\operatorname{Re} I_0 + 2 \operatorname{Re} I_1 + \operatorname{Re} I_2 \right]$$

(26)

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where

$$\left. \begin{aligned} I_0 &= \int_0^{\infty} e^{-a \cos(\theta - i\kappa)} d\kappa \\ I_1 &= \int_0^{\infty} \cosh \kappa e^{-a \cos(\theta - i\kappa)} d\kappa \\ I_2 &= \int_0^{\infty} \cosh^2 \kappa e^{-a \cos(\theta - i\kappa)} d\kappa \end{aligned} \right\} \quad (26^b)$$

and where a is the dimensionless quantity

$$a = \frac{gR}{2V^2} > 0 \quad (26^B)$$

The integration is understood to be taken along the positive real axis.

Poisson's integral representation for the zero order Hankel function (Bessel function of the third kind) $H_0^{(i)}(\zeta)$ of the complex argument ζ can be written (Ref. (1), §VIII, 5b),

$$H_0^{(i)}(\zeta) = \frac{2}{\pi} \int e^{i\zeta \cos \omega} d\omega$$

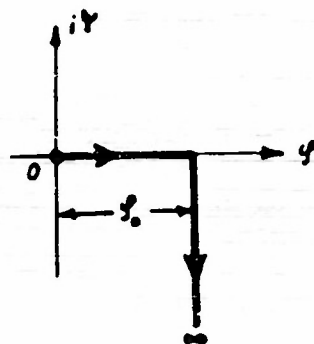
where the path of integration in the $\omega = \varphi + i\psi$ plane can be

Fig. 7

taken as indicated in Fig. 7. This is valid provided

$$-\arg \zeta < \varphi_0 < \pi - \arg \zeta$$

If we take $\zeta = ia$ and $\varphi_0 = 0$ the above condition is fulfilled

and one has

$$H_0^{(i)}(ia) = \frac{2}{\pi} \int e^{-a \cos \omega} d\omega = \frac{2}{\pi} \int_0^0 e^{-a \cos \varphi} d\varphi + \frac{2i}{\pi} \int_0^{\infty} e^{-a \cos(\theta + i\psi)} d\psi$$

Introducing $\kappa = -\psi$ in the last integral,

$$I_0 = \int_0^{\infty} e^{-a \cos(\theta - i\kappa)} d\kappa = \frac{\pi}{2} i H_0^{(i)}(ia) - i \int_0^0 e^{-a \cos \varphi} d\varphi$$

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The first term on the right is real, the second pure imaginary.

Hence,

$$\operatorname{Re} I_0 = \frac{\pi}{2} i H_0^{(1)}(ia) \quad (27)$$

independent of θ .

The integrals I_1 and I_2 can be obtained from I_0 by a recurrence formula. Consider the integral

$$I_n = \int_0^\infty \cosh^n \kappa e^{-a \cos(\theta - i\kappa)} d\kappa$$

where $n = 0, 1, 2, \dots$. Differentiating with respect to the parameter a ,

$$\frac{\partial I_n}{\partial a} = - \int_0^\infty \cosh^n \kappa e^{-a \cos(\theta - i\kappa)} (\cos \theta \sinh \kappa + i \sin \theta \cosh \kappa) d\kappa$$

whereas differentiation with respect to the parameter θ gives

$$\frac{\partial I_n}{\partial \theta} = + a \int_0^\infty \cosh^n \kappa e^{-a \cos(\theta - i\kappa)} (\sin \theta \cosh \kappa - i \cos \theta \sinh \kappa) d\kappa$$

from which, after a short calculation,

$$\frac{\sin \theta}{a} \frac{\partial I_n}{\partial \theta} - \cos \theta \frac{\partial I_n}{\partial a} = \int_0^\infty \cosh^{n+1} \kappa e^{-a \cos(\theta - i\kappa)} d\kappa$$

Hence, one has the formula

$$I_{n+1} = \frac{\sin \theta}{a} \frac{\partial I_n}{\partial \theta} - \cos \theta \frac{\partial I_n}{\partial a} \quad (28)$$

and in particular,

$$\operatorname{Re} I_{n+1} = \frac{\sin \theta}{a} \frac{\partial (\operatorname{Re} I_n)}{\partial \theta} - \cos \theta \frac{\partial (\operatorname{Re} I_n)}{\partial a} \quad (28')$$

since a and θ are real. Putting first $n=0$, after carrying out the differentiation,

$$\operatorname{Re} I_1 = - \frac{\pi}{2} \cos \theta H_1^{(1)}(ia) \quad (29)$$

With $n=1$ in Eq. (28') one finds

$$\operatorname{Re} I_2 = + \frac{\pi}{2} \left[\cos^2 \theta i H_0^{(1)}(ia) - \frac{\cos(2\theta)}{a} H_1^{(1)}(ia) \right] \quad (30)$$

With these results, from Eq. (26),

$$\frac{\partial \phi}{\partial z} = - \frac{g^2 b^2}{8 \gamma^4} e^{-a \cos \theta} \left\{ \left[(1 + \cos^2 \theta) i H_0^{(1)}(ia) - \left(2 \cos \theta + \frac{\cos(2\theta)}{a} \right) H_1^{(1)}(ia) \right] \right. \\ \left. 2b < f ; \quad x=0 \right\} \quad (31)$$

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The part of the downwash which corresponds to the bound vortex drag (Table 1) is

$$w - \frac{\partial \phi}{\partial z} = + \frac{g^2 b \bar{F}}{g V^2} \left\{ e^{-\alpha \cos \theta} \left[(1 + \cos^2 \theta) H_0^{(1)}(i\alpha) - \left(2 \cos \theta + \frac{\cos(2\theta)}{\alpha} \right) H_1^{(1)}(i\alpha) \right] - \frac{\cos(2\theta)}{\alpha^2} \right\} \quad \left. \begin{array}{l} 2b \ll f \\ x = 0 \end{array} \right\} \quad (32)$$

The terms $H_0^{(1)}(i\alpha)$ and $H_1^{(1)}(i\alpha)$ are real and are tabulated in Ref. (1) for values of α ranging from 0 to 15.9.

4. The bound vortex drag, D_B , of a short wing is

$$D_B = \frac{L}{V} \left(w - \frac{\partial \phi}{\partial z} \right)_{x=y=0}$$

where L is the total lift. With $\bar{R} = \frac{f}{2}$ and $\theta = 0$, and

introducing the Froude number \bar{F} referred to the submergence,

$$\bar{F} = \frac{V^2}{gf} \quad (33)$$

one has from Eq. (32),

$$D_B = \frac{L}{\frac{g}{2} V^2 f^2} \Psi_0(\bar{F}) \quad ; \quad 2b \ll f$$

or also

$$\frac{D_B}{L} = C_L \frac{S}{f^2} \Psi_0(\bar{F}) \quad (34)$$

where

$$\Psi_0(\bar{F}) = \frac{1}{32} \left\{ \frac{e^{-\frac{1}{\bar{F}}}}{\bar{F}^2} \left[2 H_0^{(1)}\left(\frac{1}{\bar{F}}\right) - (2 + \bar{F}) H_1^{(1)}\left(\frac{1}{\bar{F}}\right) \right] - \frac{1}{\pi} \right\}$$

A similar result has been obtained by Havelock for the case of a submerged sphere, Ref. (f).

In the particular case of a very small Froude number, the term which contains the exponential and Hankel functions tends towards zero for $\bar{F} \rightarrow 0$ and one obtains

$$(\Psi_0)_{\bar{F} \rightarrow 0} = - \frac{1}{32\pi} = -.00995 \quad (34^I)$$

The bound vortex drag over lift ratio is negative in this case.

The same result can be derived also in the following manner: Since the Froude number is a measure of the ratio of inertia force and gravity force, gravity becomes increasingly important as compared with inertia if $\bar{F} \rightarrow \infty$. It follows then from Eq. (3) that

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$$x_0 = 0$$

i. e. the free surface acts like a rigid boundary in this case (Ref. (b)). The boundary condition at the surface can then be satisfied through the introduction of the image system (3) of section 3.1 alone, without an additional perturbation potential. All that is left in Eq. (34) is the term $\frac{1}{\pi}$ which originated from the velocity induced by the image system.

For large values of F , i.e. small arguments of the Hankel functions, one has the asymptotic values *)

$$\begin{aligned} i H_0^{(1)}\left(\frac{i}{F}\right) &= + \frac{2}{\pi} \log_e \left(\frac{2}{F}\right) \\ H_1^{(1)}\left(\frac{i}{F}\right) &= - \frac{2}{\pi} F \end{aligned}$$

($\gamma =$ Euler's constant). From Eq. (34), one obtains then after short calculation,

$$\left(\frac{\gamma}{\pi}\right)_{F \rightarrow \infty} = + \frac{1}{32 \cdot \pi} = + .00995 \quad (34'')$$

Gravity being negligible as compared with inertia forces in this case, the boundary condition at the surface can be satisfied by means of an image system of opposite circulation as compared with the image system employed so far. This consideration leads also, independently, to Eq. (34'). The bound vortex drag in this case is identical with the "mutually induced drag" of a biplane.

The function γ_0 is plotted in Fig. 8**). The bound vortex drag is found to vanish at a Froude number $F = 0.64$.

It will be shown in section 7 that the condition $2b \ll f$ so far imposed, can be greatly relaxed without invalidating Eq. (34). For instance,

*) See also Ref. (a) where this approach has been first suggested.

**) A similar curve is presented in Ref. (a), where, however, a somewhat different definition for the dimensionless function γ_0 is employed.

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$$\begin{aligned}
 \text{or also} \quad D_s &= \frac{L^2}{\rho/2 V^2} \cdot \frac{\mu}{2bf} \cdot \psi_1(F) \quad ; \quad \begin{matrix} 2b \gg f \\ 2b \gg \frac{V^2}{g} \end{matrix} \\
 \frac{D_s}{L} &= C_L \frac{S}{2bf} \mu \psi_1(F) \\
 \text{where} \quad \psi_1(F) &= \frac{s - \frac{1}{F}}{2F} \\
 \text{and} \quad \mu &= \frac{\overline{L^2}}{(\overline{L})^2}
 \end{aligned} \quad (35)$$

In the last equation, the symbol $\overline{\quad}$ means the span-wise average of the quantity in question. μ depends upon the lift distribution only. Whereas, for a rectangular lift distribution, $\mu = 1$, one finds easily for an elliptic distribution,

$$\mu = \frac{32}{3\pi^2} = 1.08$$

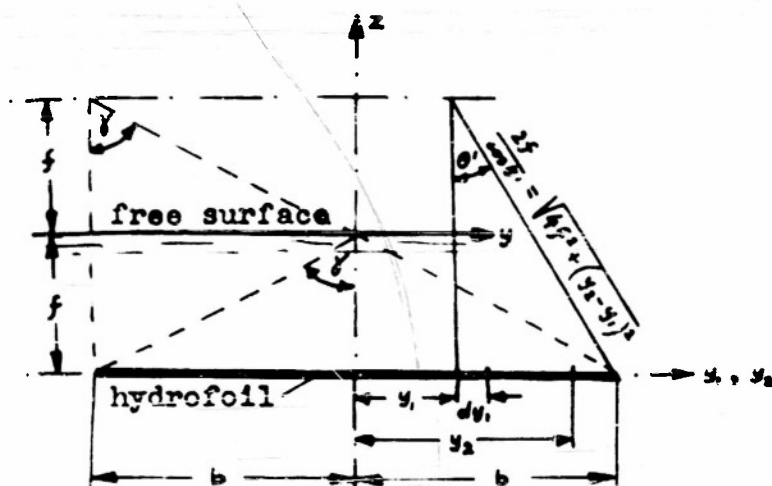
The function $\psi_1(F)$ is also plotted in Fig. 8. $\psi_1(F)$ vanishes for $F = 0$ and for $F = \infty$ as one recognizes from the expression found for ψ_1 . The physical argument utilized in section 5.4 for the cases $F = 0$ and $F = \infty$, leads also to the conclusion that the bound vortex drag must vanish in these cases.

7. General Case of the Wave Drag of a Single Hydrofoil

1. Consider now the part of the downwash which corresponds to the bound vortex drag and which is induced at $x = 0, y = y_1, z = -f$ by a wing element dy_1 , located at $x = 0, y = y_2, z = -f$ (Fig. 9).

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Fig. 9



For this purpose, one has to replace in Eq. (32) $2b$ by dy , and a by

$$a = \frac{g}{2V^2} \frac{2f}{\cos \theta'} = \frac{1}{F \cos \theta'}$$

Hence, one obtains for the downwash angle, i.e. the ratio of the downwash velocity and the speed of advance,

$$\text{where } \frac{L_1(y_1)}{\rho/2 V^2 f^2} \psi_2(F, \theta') dy_1$$

$$\psi_2(F, \theta') = \frac{1}{32} \left\{ \frac{e^{-\frac{1}{2}}}{F^2} \left[(1 + \cos^2 \theta') i H_0^{(1)} \left(\frac{i}{F \cos \theta'} \right) - (2 \cos \theta' + F \cos \theta' \cos(2\theta')) H_1^{(1)} \left(\frac{i}{F \cos \theta'} \right) \right] \right. \\ \left. - \frac{\cos^2 \theta' \cos(2\theta')}{\pi} \right\} \quad (36)$$

and where

$$\theta' = \arctan \frac{y_2 - y_1}{2f}$$

In Fig. 10, ψ_2 is plotted versus the dimensionless distance

$$\frac{y_2 - y_1}{f} = 2 \tan \theta'$$

for three different values of the Froude number,

$$F = 0$$

$$F = 10.0$$

$$F = \infty$$

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In the case $F = 0$, the terms containing the exponential and Hankel functions vanish. Consequently,

$$(\Psi_1)_{F=0} = - \frac{\cos^2 \theta' \cos(2\theta')}{32 \pi} \quad (36^I)$$

In the case $F = \infty$, making use of the asymptotic representation for the Hankel functions,

$$(\Psi_1)_{F=\infty} = + \frac{\cos^2 \theta' \cos(2\theta')}{32 \pi} \quad (36^{II})$$

The right-hand side of Eqs. (36^I) and (36^{II}) vanishes for $\theta' = \pm \frac{\pi}{4}$ corresponding to $\frac{y_2 - y_1}{f} = \pm 2$. At these points the downwash changes its sign.

2. The total downwash angle (as far as the bound vortex drag is concerned) at $y = y_2$ is

$$\frac{1}{\rho/2 V^2 f^2} \int_{y_1=-b}^{y_2=b} L_1(y_1) \Psi_2 \left(F, \pi \tan \frac{y_2 - y_1}{2f} \right) dy_1, \quad (37)$$

and from this the bound vortex drag,

$$D_B = \frac{1}{\rho/2 V^2 f^2} \int_{y_2=-b}^{y_2=b} \int_{y_1=-b}^{y_1=b} L_1(y_1) L_2(y_2) \Psi_2 \left(F, \pi \tan \frac{y_2 - y_1}{2f} \right) dy_2 dy_1,$$

Introducing the dimensionless quantities $\lambda_1 = y_1/f$, $\lambda_2 = y_2/f$ and the span-submergence ratio

$$\beta = \frac{2b}{f} \quad (38)$$

one has

$$D_B = \frac{L^2}{\rho/2 V^2} \cdot \frac{1}{2bf} \Psi_3$$

or also

$$\begin{aligned} \frac{D_B}{L} &= C_L \frac{S}{2bf} \Psi_3 \\ \text{There is} \quad \Psi_3 &= \frac{1}{\beta} \int_{\lambda_2=-\beta/2}^{+\beta/2} \int_{\lambda_1=-\beta/2}^{+\beta/2} \frac{L_1(f\lambda_1)}{L_1} \cdot \frac{L_2(f\lambda_2)}{L_2} \cdot \Psi_2 \left(F, \pi \tan \frac{\lambda_2 - \lambda_1}{2} \right) d\lambda_2 d\lambda_1 = \\ &= \frac{1}{32\beta} \int_{\lambda_2=-\beta/2}^{+\beta/2} \int_{\lambda_1=-\beta/2}^{+\beta/2} \frac{L_1(f\lambda_1)}{L_1} \cdot \frac{L_2(f\lambda_2)}{L_2} \left\{ \frac{e^{-\frac{1}{F}}}{F^2} \left[\left(1 + \cos^2 \theta' \right) H_0^{(1)} \left(\frac{i}{F \cos \theta'} \right) - \left(2 \cos \theta' + F \cos \theta' \cos(2\theta') \right) H_2^{(1)} \left(\frac{i}{F \cos \theta'} \right) \right] \right. \\ &\quad \left. - \frac{\cos^2 \theta' \cos(2\theta')}{\pi} \right\} d\lambda_2 d\lambda_1 \end{aligned} \quad (39)$$

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where

$$\theta' = \arctan \frac{\lambda_2 - \lambda_1}{2}$$

Besides depending from F and β , Ψ_2 depends also from the lift-distribution. If one puts

$$L_1 = L_{1, \max} \left[1 - \left(\frac{y}{b} \right)^2 \right]^n$$

the distribution is

rectangular for $n = 0$

elliptic for $n = 1/2$

paraboloid for $n = 1$

The total lift $L = 2b\bar{L}_1$ is found

$$L = L_{1, \max} \int_{y=-b}^{+b} \left[1 - \left(\frac{y}{b} \right)^2 \right]^n dy = 2bL_{1, \max} \int_0^{\pi/2} \cos^{2n+1} \nu d\nu = \sqrt{\pi} \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} bL_{1, \max}$$

after short calculation, where Γ signifies here the gamma function.

Consequently, one can replace in this case $\frac{L_1(f\lambda_1)}{\bar{L}_1} \cdot \frac{L_1(f\lambda_2)}{\bar{L}_1}$ in Eq. (39) by

$$\frac{L_1(f\lambda_1)}{\bar{L}_1} \cdot \frac{L_1(f\lambda_2)}{\bar{L}_1} = \frac{4\Gamma^2(n+\frac{1}{2})}{\pi\Gamma^2(n+1)} \left\{ \left[1 - \left(\frac{2\lambda_1}{\beta} \right)^2 \right] \left[1 - \left(\frac{2\lambda_2}{\beta} \right)^2 \right] \right\}^n \quad (39^1)$$

For instance, for $n = 1/2$ (elliptic distribution) this becomes,

$$\frac{L_1(f\lambda_1)}{\bar{L}_1} \cdot \frac{L_1(f\lambda_2)}{\bar{L}_1} = \frac{16}{\pi^2} \sqrt{\left[1 - \left(\frac{2\lambda_1}{\beta} \right)^2 \right] \left[1 - \left(\frac{2\lambda_2}{\beta} \right)^2 \right]}$$

In contrast to the integral encountered in calculating the induced drag (trailing vortex drag), the bound vortex drag (Eq. (39)) is expressed by a proper integral. It can be evaluated for instance with the aid of a planimeter:

In Fig. 11 Ψ_2 is plotted versus β for $F = 10.85$ and $n = 0$.

It is convenient, in this case, to take as new variables of integration in Eq. (39),

$$\Lambda_1 = \lambda_2 - \lambda_1$$

$$\Lambda_2 = \lambda_2$$

Consequently,

$$\Psi_2 = \frac{1}{\beta} \int_{\Lambda_2 = \frac{\beta}{2}}^{\Lambda_2 = \frac{\beta}{2} + \Lambda_1} \int_{\Lambda_1 = \Lambda_2 - \frac{\beta}{2}}^{\Lambda_1 = \Lambda_2 - \frac{\beta}{2}} \Psi_2 \left(F, \arctan \frac{\Lambda_1}{\Lambda_2} \right) d\Lambda_1 \quad ; \quad n=0 \quad (39^2)$$

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Y_3 is then obtained by integrating under always the same curve Y_1 in Fig. 10, but for various values of the limits of integration; plotting the result of the integration versus Λ_2 and integrating once more gives Y_3 .

3. If expression (34) for the bound vortex drag of a short wing is written in the same form as Eq. (39), one obtains,

$$Y_3 = \beta Y_1$$

in this case. The straight lines corresponding to this relationship are also drawn in Fig. 11, for $F = 0, 10.85$ and ∞ . They are tangent to the curves plotted in this figure. The slope reaches a maximum at about $F = 2.5$ (Fig. 8) and decreases then again for increasing Froude number. Hence, part of Fig. 11 is doubly covered by the lines $F = \text{constant}$.

Similarly one has for a long wing,

$$Y_3 = Y_1$$

This relationship corresponds to horizontal lines, to which the curves in Fig. 11 are asymptotic.

The bound vortex drag is independent of the lift distribution in the case of a short wing (Eq. (34)), and is, for practical purposes, almost independent from it in the case of a long wing (section 6). Presumably this is therefore true for the whole range of β . This fact makes the distinction between bound vortex drag and trailing vortex drag very convenient for calculating purposes: the former depends upon the Froude number and the span-submergence ratio, but little upon the lift distribution; the latter depends only upon the lift distribution.

4. In the case $F = 0, n = 0$, Eq. (39^{II}) simplifies to

$$Y_3 = - \frac{1}{32\pi\beta} \int_{\Lambda_2 = -\beta_2}^{+\beta_2} d\Lambda_2 \int_{\Lambda_1 = \Lambda_2 - \beta_2}^{\Lambda_2 + \beta_2} \cos^2 \theta' \cos(2\theta') d\Lambda_1$$

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The second integral is with $\theta' = \arctan \frac{\Lambda_1}{\beta}$
 $\int_{\Lambda_1 = \Lambda_2 - \frac{\beta}{2}}^{\Lambda_1 = \Lambda_2 + \frac{\beta}{2}} \frac{1 - \frac{\Lambda_1^2}{\beta^2}}{(1 + \frac{\Lambda_1^2}{\beta^2})^2} d\Lambda_1 = \frac{\Lambda_1}{1 + \frac{\Lambda_1^2}{\beta^2}} \Big|_{\Lambda_1 = \Lambda_2 - \frac{\beta}{2}}^{\Lambda_1 = \Lambda_2 + \frac{\beta}{2}}$
 Integrating over Λ_1 , one finds after short calculation,

$$\Psi_3 = - \frac{1}{8\pi} \cdot \frac{\log_e (1 + \frac{\beta^2}{\gamma^2})}{\beta} ; \quad F \ll 1, \quad n=0 \quad (39^{III})$$

This expression for Ψ_3 may be regarded as an approximation in the case of a rectangular lift distribution and of a very small Froude number, and is plotted in Fig. 11.

The surface acting in this case like a rigid sheet, one can derive Eq. (39^{III}) also directly by considering the downwash Eq. (25) produced by the appropriate image.

5. θ' in Eq. (39) is never larger than γ , where γ is the half-angle under which the span appears if viewed from the midpoint on the surface (Fig. 9). Provided that

$$F \cos \gamma = \frac{F}{\sqrt{1 + \frac{\beta^2}{\gamma^2}}} \gg 1$$

the argument of the functions $H_0^{(1)}$ and $H_1^{(1)}$ in Eq. (39) becomes very small. Making use of the asymptotic representation for the Hankel

functions and assuming $n=0$, $\Psi_3 = + \frac{1}{32\pi\beta} \int_{\Lambda_1 = -\frac{\beta}{2}}^{+\frac{\beta}{2}} d\Lambda_1 \int_{\Lambda_2 = \Lambda_1 - \frac{\beta}{2}}^{\Lambda_2 = \Lambda_1 + \frac{\beta}{2}} \cos 2\theta' \cos(2\theta') d\Lambda_2$,
 which is the same as obtained in the case $F=0$, except for the sign.

Consequently,

$$\Psi_3 = + \frac{1}{8\pi} \cdot \frac{\log_e (1 + \frac{\beta^2}{\gamma^2})}{\beta} ; \quad F \cos \gamma \gg 1, \quad n=0 \quad (39^{IV})$$

For instance, for $F = 10.85$ and $\gamma = 75^\circ$ (corresponding to $F \cos \gamma = 7.67$) one underestimates the bound vortex drag by 22% if this equation is used as a first approximation.

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6. For comparison, consider the trailing vortex (induced) drag

$$D_T = \frac{L^2}{\rho \frac{1}{2} V^2} \cdot \frac{1}{4\pi b^2}$$

for an elliptic lift distribution (e.g. Ref. (e)). For easier comparison with Eq. (39), one may write the above equation,

$$D_T = \frac{L^2}{\rho \frac{1}{2} V^2} \cdot \frac{1}{2bf} \cdot \frac{1}{\pi/\beta} \quad ; \quad n = \frac{1}{2} \quad (40)$$

The expression $\frac{1}{\pi/\beta}$ is also plotted in Fig. 11.

The trailing vortex drag predominates over the bound vortex drag in the case of a sufficiently small span. It follows then from the corresponding theorem derived in airplane wing theory, that the optimum lift distribution (i.e. the distribution for which the total drag of a single foil is a minimum) is elliptic.

On the other hand, for a sufficiently large span, the bound vortex drag becomes predominant. It follows then from Eq. (35) that the optimum lift distribution approaches a rectangular distribution. *)

8. Appendix

If one introduces in Eq. (39) instead of the variables of integration λ_1 and λ_2 ,

$$\eta_1 = 2\lambda_1$$

$$\eta_2 = 2\lambda_2$$

and with the symbols

$$u = \frac{1}{F \cos \theta'} = \frac{1}{F} \sqrt{1 + \left(\frac{\lambda_2 - \lambda_1}{2}\right)^2} = \frac{1}{F} \sqrt{1 + \left(\frac{\eta_2 - \eta_1}{4}\right)^2} \quad (A-1)$$

$$v = \frac{\sqrt{\cos(2\theta')}}{F \cos \theta'} = \frac{1}{F} \sqrt{1 - \left(\frac{\lambda_2 - \lambda_1}{2}\right)^2} = \frac{1}{F} \sqrt{1 - \left(\frac{\eta_2 - \eta_1}{4}\right)^2} \quad (A-2)$$

*) μ in Eq. (35) cannot be smaller than 1. This latter value is obtained for a rectangular lift-distribution.

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one obtains after short calculation,

$$D_s = \frac{L_1 \text{ mas}}{64 \pi V^2 F_1} \int_{-\beta}^{\beta} \int_{-\beta}^{\beta} \left\{ \left[1 - \left(\frac{\eta_1}{\beta} \right)^2 \right] \left[1 - \left(\frac{\eta_2}{\beta} \right)^2 \right] \right\}^n \left\{ e^{-\frac{1}{2}} \left[\left(1 + \frac{1}{F_1 u^2} \right) H_0^{(1)}(iu) - \frac{1}{u} \left(\frac{2}{F} + \frac{v^2}{u^2} \right) H_1^{(1)}(iu) \right] - \frac{v^2}{\pi u^4} \right\} d\eta_1 d\eta_2 \quad (A-3)$$

This is the expression for the bound vortex drag stated in the memorandum

Ref. (j) containing the advance information on the present subject material.

R. X. Meyer

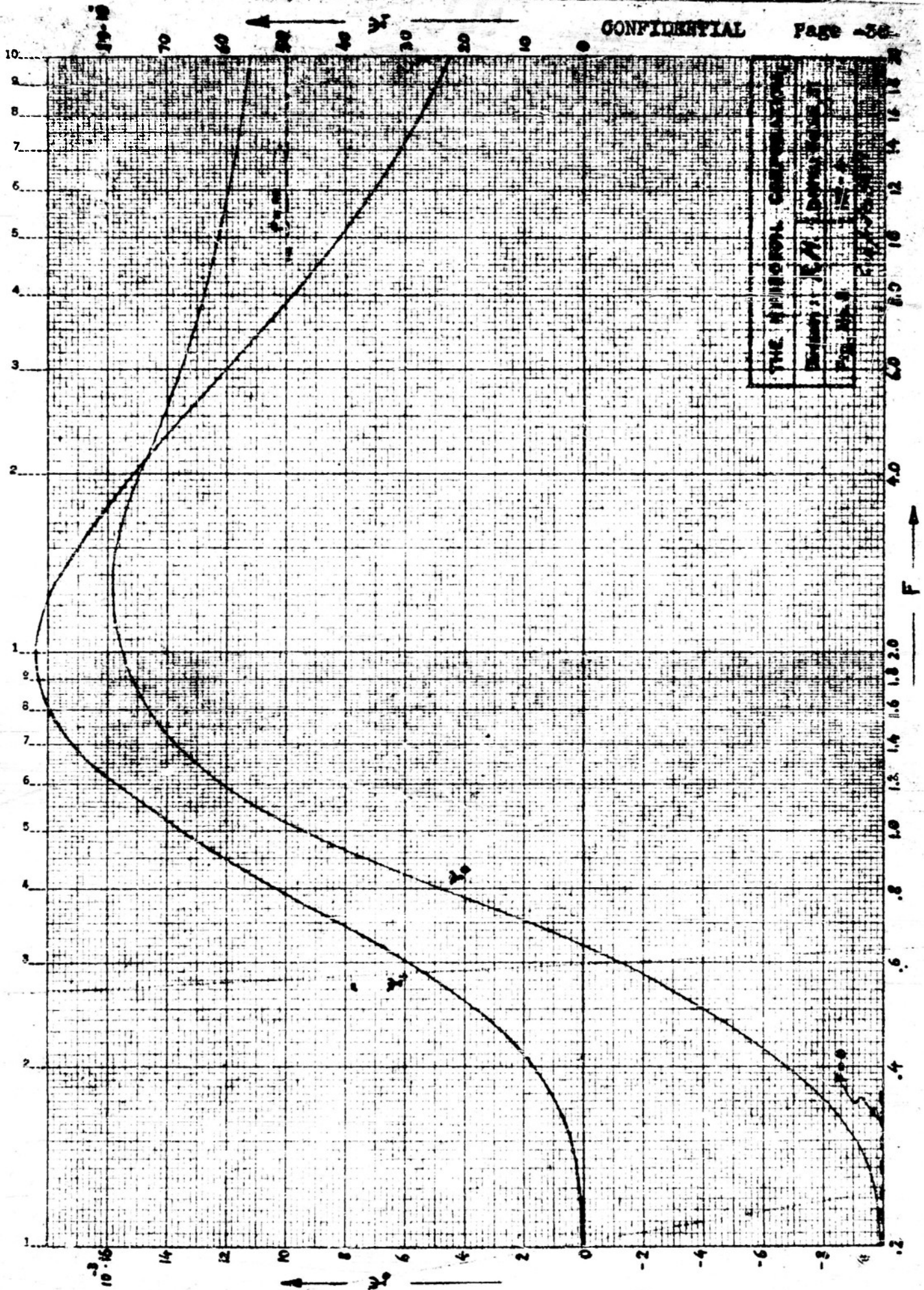
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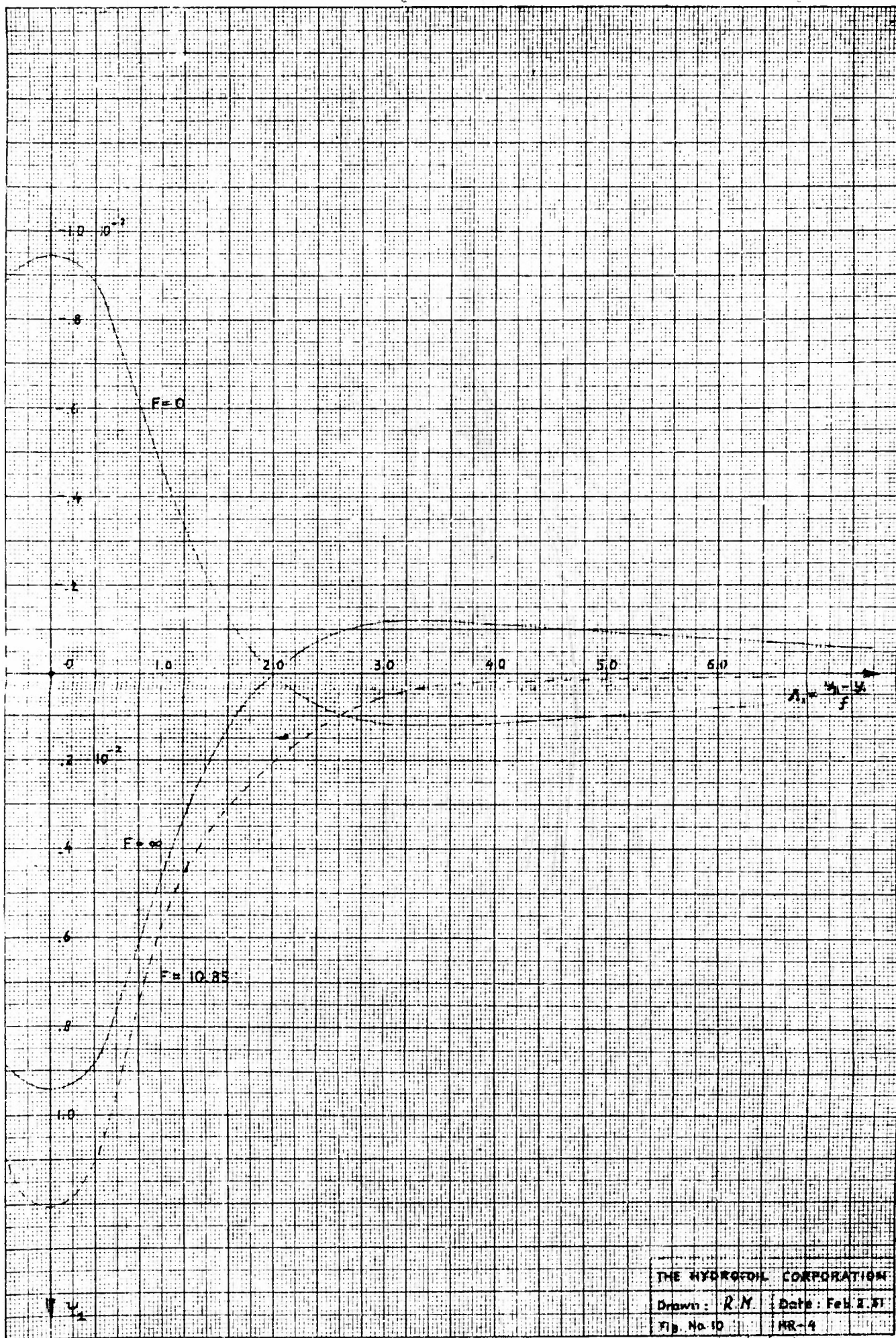
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$W_2(r, \theta, n)$

$r = 0.95, 2b \ll r$ (EQ 59)
 $r = 1, 2b \ll r$ (EQ 60)
 $r = 1.05, 2b \ll r$ (EQ 61)

$r = 1.05, n = 0$ (EQ 59)

$r = 1.05, n = 0$ (EQ 61)

$r = 1, n = 0$ (EQ 60)

$r = 1, 2b \ll r$ (EQ 62)

$r = 0.95, n = 1, 2b \ll r$ (EQ 59)

$r = 1.05, n = 0, 2b \ll r$ (EQ 61)

ASYMPT.

$r = 1, 2b \ll r$ (EQ 60)

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Fig. No. II	HR. 4	